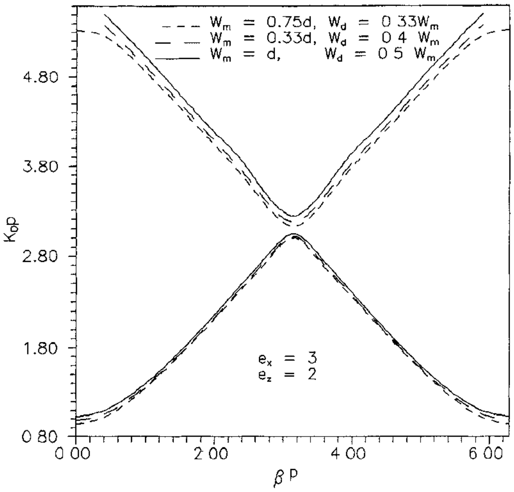


Fig. 3. Dispersion characteristic for sinusoidal slot variation.

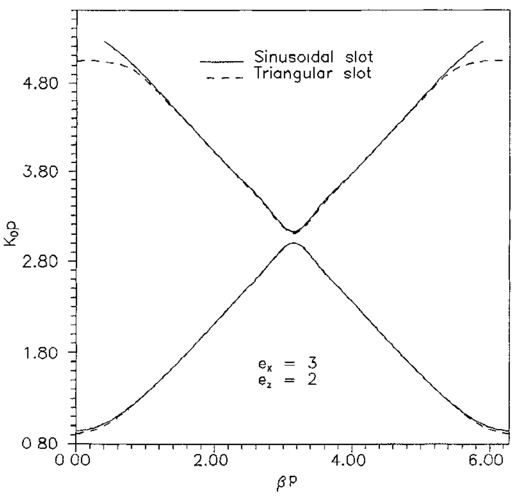


Fig. 4. Dispersion characteristics for sinusoidal and triangular slot variation.

stopband of the sinusoidally varying slot pattern. This may be due to broader apex of the sinusoidal slot in comparison to sharper apex of the triangular slot.

## IV. Conclusion

The SSDA has been extended for computing the dispersion characteristics of some periodic structures in the $k_0 - \beta$ plane. With the incorporation of the periodic boundary conditions, the present method is very well suited to analyze various possible periodic structures in microstrip, fin lines, and co-planar waveguides, which were difficult to analyze before. The interesting feature of the method is that the same set of the sinusoidal basis functions can be utilized for most of slot geometry. This method can also be utilized to study other types of periodic structures, like meander lines and so on.

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# Complex Solitons in a Superconductive Medium

K. Hayata and M. Koshiba

***Abstract*—We show analytically that a type-II superconductor may support short-range electromagnetic spatial solitons with a complex propagation constant. A theoretical model based on the Ginzburg-Landau theory is used. Analytical results for the complex solitons predict unique features that cannot be found in conventional solitons in normal (asuperconductive) media.**

## I. Introduction

Soliton and solitary-wave propagation in material media such as dielectrics, semiconductors, plasmas, and magnetized materials have long been of extensive interest in a rich variety of branches that include both pure and applied sciences [1]. For solitary light beams (spatial solitons) that are describable with a family of nonlinear Schrödinger equations, a picture that explains solitons in terms of the fundamental modes of the linear waveguide they induce was found to be consistent with our physical intuition [2], [3]. As is well known in classical waveguide theory, guided modes in a linear waveguide can be classified into three types: bound (oscillatory), evanescent (diffusive), and complex modes, which can be characterized by a real, a purely imaginary, and a complex propagation constant, respectively.

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The bound modes are normal guided modes, which propagate in a waveguide without attenuation as long as material absorption is negligible. In striking contrast to these, the evanescent modes decay exponentially along the propagation axis without sinusoidal oscillations, even if the absorption would vanish. For instance, one can observe an evanescent mode in the lower vicinity of the cutoff frequency of a commercially available waveguide used for microwave transmission. The complex modes, for which neither the phase nor the attenuation constant vanishes, are known to exist in several lossless waveguides, such as dielectric-loaded waveguides [4]–[6], finlines [7], and microstrip lines [8]. As mentioned above, it is obvious that solitons, termed in the usual context, are closely related with the bound modes [3]. In the similar context, we would like to pose an interesting question: Is there a solitonic entity that has analogy to the complex and the evanescent modes of a linear waveguide? If so, how is that expressed algebraically? One will notice immediately that the answer is quite nontrivial. Indeed, to our knowledge there has been no report in which an example of such an analog is presented. Our conclusion is that, at least for such normal media as cited above, one could not find a candidate of the complex soliton (i.e., the soliton with complex propagation constant). In this paper, we find analytically that a type-II superconductor may support a solitonic analog of the complex modes, which approaches the evanescent soliton in the limit of low frequency. In the framework of the Ginzburg-Landau (GL) theory [9], analytical results for high-frequency electromagnetic wave propagation in the form of a spatial complex soliton are presented.

## II. Formalism Based on Ginzburg-Landau Theory

We consider the time-harmonic electromagnetic wave propagation in bulk superconductors. We assume that the energy of an electromagnetic quantum (i.e., a photon) is much lower than the gap energy ($\hbar\omega \ll \Delta$), and the temperature is considerably lower than the critical value ($T \ll T_c$). In such situations the contribution from the frequency-dependent normal (asuperconducting) current $|\underline{j}_n(\omega)|$ will be sufficiently smaller than that from the superconducting current $|\underline{j}_s|$. This relation, $|\underline{j}_n(\omega)| \ll |\underline{j}_s|$, allows one to employ a perturbational treatment based on quasi-static approximation [9], [10], and thus leads to the GL equations

$$\xi^2[i\nabla + (2\pi/\phi_0)\underline{A}]^2\Psi - \Psi + |\Psi|^2\Psi = 0 \tag{1a}$$

$$\nabla \times \nabla \times \underline{A} = (4\pi/c)[\underline{j}_s + i\underline{j}_n(\omega)] \tag{1b}$$

$$\underline{j}_s(\underline{r}) = -i[en_s(T)/(4m)](\Psi^* \cdot \nabla\Psi - \nabla\Psi^* \cdot \Psi) - [e^2/(mc)]n_s(T)|\Psi|^2\underline{A} \tag{1c}$$

where $\Psi(\underline{r})$ is the order parameter [$\underline{r}$ is a vector that indicates spatial dependence of field, i.e., for Cartesian coordinate, $\underline{r} = (x, y, z)$], $\underline{A}(\underline{r})$ is the vector potential, $\xi$ is the coherence length, $\phi_o$ is the flux quantum, and $n_s(T)$ stands for the density of the superconductive component. Other symbols $(c, e, m)$ obey the usual definition in the GL theory [9]. This theory is established as a representative phenomenological approach to studying superconducting states that undergo spatial variations due to an external field. In (1b) $\underline{j}_s(\underline{r})$ [$\underline{j}_n(\underline{r})$] indicates the current density vector due to superconductivity (normal conductivity); the argument "$\omega$" of the normal current has been attached to indicate explicitly that it depends significantly on the frequency of the time-harmonic electromagnetic field. The explicit form of $\underline{j}_n$ depends not only on the frequency but on the temperature under consideration and the purity of a superconductor [see (3)–(5)]. Note that for the field variables in (1), the time-harmonic dependence of $\exp(-i\omega t)$ is implied.

The specific form of $\underline{j}_n$ can be written as [9]

$$\underline{j}_n = Q(\omega)\underline{A} \tag{2a}$$

with

$$Q(\omega) = (3/4)[(n_e e^2)/(mc)][\Delta/(\hbar vq)]Q_1(\omega) \tag{2b}$$

where $n_e$ is the number density of electrons and $q$ is the center-of-mass momentum of a Cooper pair. The expression of $Q_1$ depends on the temperature. At $T = 0$, superconductors do not contain quasi-particles that could absorb quanta of any energy. The absorption of electromagnetic waves occurs when $\hbar\omega \geq 2\Delta$. In the vicinity of the threshold we obtain [9]

$$Q_1 \sim \pi^2\{(\hbar\omega)/[2\Delta(0)] - 1\}. \tag{3}$$

Note that for $\hbar\omega \gg 2\Delta$ the difference between a superconductor and a normal metal disappears. The characteristic wavelengths corresponding to the threshold ($\hbar\omega = 2\Delta$) lie within a range of the order of 1–0.1 mm and the corresponding frequencies of the order of $10^{11}$–$10^{12}$ Hz. At $T > 0$ there exist quasi-particles in the superconductor that can absorb photons of any frequency. The expression that is valid for $\omega \ll \Delta$, $T \ll \Delta$ is given by [9]

$$Q_1 \sim 4\pi \sinh[\omega/(2T)]K_0[\omega/(2T)]\exp[-\Delta(0)/T] \tag{4}$$

where $K_0$ is the modified Bessel function of the second kind. For a dirty superconductor in the London limit near $T_c$, when $\omega \ll \Delta \ll T_c$ (obviously, for so-called high-$T_c$ superconductors this condition would become much relaxed), we have the following relation

$$Q(\omega) = \omega\sigma/c \tag{5}$$

where $\sigma$ is the conductance due to the superconductor alloy.

## III. Complex Soliton Solutions

Concentrating on the transverse-electric (TE) polarized field (where $\partial/\partial y \equiv 0$) and on a gauge $\underline{A} = (0, A, 0)$, (1) with (2a) can be reduced in the form

$$\kappa^{-2}(\partial^2/\partial x^2 + \partial^2/\partial z^2 - A^2)\Psi + \Psi - \Psi^3 = 0 \tag{6a}$$

$$(\partial^2/\partial x^2 + \partial^2/\partial z^2)A + i\gamma A - \Psi^2 A = 0 \tag{6b}$$

where $\Psi$ has been assumed to be real. With this assumption, the first term on the right-hand side of (1c) is dropped, which results in the substantial reduction of subsequent algebraic effort. In the derivation of (6), the physical variables have been scaled according to

$$x \to x/\lambda,\ z \to z/\lambda,\ \beta \to \beta\lambda,\ A \to A/[\phi_0/(2\pi\lambda)]. \tag{7}$$

With this normalization, $\kappa$ and $\gamma$ are defined as

$$\kappa = \lambda/\xi,\ \gamma = 2\phi_0 Q/(c\lambda) \tag{8}$$

both of which must be positive.

For an explicit form of the total vector field $(\Psi, A)$, we consider

$$\Psi(x, z) = \psi(x) \tag{9a}$$

$$A(x, z) = (1/2)A(x)\exp(i\beta z) + \text{c.c.} \tag{9b}$$

where $\beta$ is the propagation constant (we define $\beta' \equiv \mathrm{Re}\,\beta > 0$, $\beta'' \equiv \mathrm{Im}\,\beta > 0$) and c.c. denotes complex conjugate. On substitution of (9) into (6), one obtains a set of simultaneous nonlinear ordinary differential equations with three unknowns $[A(x), \psi(x), \beta]$

$$d^2\psi/dx^2 + \kappa^2\psi - \kappa^2\psi^3 \cong 0 \tag{10a}$$

$$d^2A/dx^2 + (i\gamma - \beta^2)A - \psi^2 A = 0 \tag{10b}$$

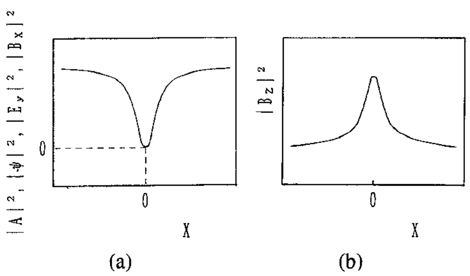


Fig. 1. Schematic transverse intensity profile of component fields of complex soliton in a superconductive medium. (a) Dark component. (b) Bright component.

where $\kappa$ is the GL parameter that is defined in (8) and $\gamma$ is a frequency-dependent real parameter that comes from the normal current in (1c); the specific form of $\gamma$ is determined from (2)–(5). In the derivation of (10a), we have dropped a term proportional to $|A|^2\psi$, implying that $|A_0| \ll \kappa|\psi_0|$. (The definition of $A_0$ and $\psi_0$ is presented below.) Because (10) are coupled nonlinear Schrödinger-type equations with self-defocusing nonlinearity, it will be reasonable to set a particular ansatz of a dark soliton [1]

$$\psi(x) = \psi_0 \tanh(\alpha x),\ A(x) = A_0 \tanh(\alpha x) \tag{11}$$

where $(\alpha,\ \psi_0,\ A_0)$ are real parameters that feature the soliton profile shown in Fig. 1(a). Note that it is the addition of the term $\psi^2 A$ in (10b) that is responsible for inducing the nontrivial transverse pattern of the vector potential in an unbounded medium. In the framework of our perturbational approach based upon a quasi-static treatment, it must be maintained that $\gamma \ll \psi_0^2$ [10]. On substitution of (11) into (10), to be self-consistent, it must be required that

$$\alpha = 2^{-1/2},\ |\psi_0| = 1,\ \kappa = 1 \tag{12}$$

$$\beta = \beta' + i\beta'' = (i\gamma - 1)^{1/2}. \tag{13}$$

Note that the value of $\kappa$ exceeds $\kappa_c \equiv 2^{-1/2} \fallingdotseq 0.7071$, which is the critical value between the type-I $(0 < \kappa < \kappa_c)$ and the type-II $(\kappa > \kappa_c)$ superconductors [9]. From (13) one can obtain the explicit form of the propagation constant

$$\beta' = \gamma\{2[1 + (1 + \gamma^2)^{1/2}]\}^{-1/2} \tag{14a}$$

$$\beta'' = \{[1 + (1 + \gamma^2)^{1/2}]/2\}^{1/2} \tag{14b}$$

where $\gamma \ll 1$. It should be noted in (14) that as the complex modes that were predicted for microwave transmission lines [4]–[8], the magnitude of the attenuation constant $\beta''$ is comparable to that of the phase constant $\beta'$. In the limit of $\gamma \rightarrow 0$, (14) predict $\beta' \rightarrow 0$, $\beta'' \rightarrow 1$, which shows a typical feature of the evanescent soliton in the sense that the phase constant vanishes. The evanescent soliton has analogy to the waveguide mode below a cutoff frequency. The penetration depth of the complex soliton (the soliton with complex propagation constant) is estimated by $L \equiv 1/\beta''$. For instance, for $\gamma = 10^{-2}$, $L = 1.0$.

As an alternative for (9a) in what follows, we consider

$$\Psi(x,\ z) = (1/2)\psi(x) \exp(\mathrm{ipz}) + \mathrm{c.c.} \tag{15}$$

where $p$ is a nonvanishing real parameter that indicates the pitch of a sinusoidal modulation along the longitudinal (the $z$) axis. With vanishing $p$, the longitudinal variation of the order parameter disappears, and apparently (15) is reduced to (9a). On substitution of (9b) and (15) into (6), as in the case of $p = 0$, one obtains a set of simultaneous nonlinear ordinary differential equations with the three unknowns $[A(x),\ \psi(x),\ \beta]$

$$d^2\psi/dx^2 + \kappa^2\psi - (3/4)\kappa^2\psi^3 \cong 0 \tag{16a}$$

$$d^2A/dx^2 + (i\gamma - \beta^2)A - (1/2)\psi^2 A \cong 0. \tag{16b}$$

Here, rotating wave approximation that will be valid for $\beta' \ll |p|$ has been applied, and as in the derivation of (10), implying that $|A_0| \ll \kappa|\psi_0|$, we have dropped the term proportional to $|A|^2\psi$. Substituting (11) into (16), we derive

$$\alpha = 3^{-1/2},\ |\psi_0| = (4/3)^{1/2},\ \kappa = (2/3)^{1/2} \tag{17}$$

$$\beta = \beta' + i\beta'' = (i\gamma - 2/3)^{1/2} \tag{18}$$

with

$$\beta' = (\gamma/2)\{6/[2 + (4 + 9\gamma^2)^{1/2}]\}^{1/2} \tag{19a}$$

$$\beta'' = \{[2 + (4 + 9\gamma^2)^{1/2}]/6\}^{1/2}. \tag{19b}$$

From (17), in the present case as well, the GL parameter is larger than the critical value $[\kappa = (2/3)^{1/2} \fallingdotseq 0.8165 > \kappa_c]$. In the limit of $\gamma \rightarrow 0$, it follows that $\beta' \rightarrow 0$, $\beta'' \rightarrow (2/3)^{1/2}$. Again one can find the evanescent soliton. For instance, for $\gamma = 10^{-2}$, $L = 1.22$.

## IV. Electromagnetic-Field Components

From (9b) and (11), the expression of the total vector potential can be written in the form

$$A(x,\ z,\ t) = |A_0| \tanh(\alpha x) \exp(-\beta'' z) \cos(\beta' z - \omega t). \tag{20}$$

Thus, the resultant nonvanishing electromagnetic-field components are straightforwardly derivable

$$E_y = -\partial A/\partial t = -\omega|A_0| \tanh(\alpha x) \exp(-\beta'' z) \sin(\beta' z - \omega t) \tag{21a}$$

$$B_x = -\partial A/\partial z = |A_0| \tanh(\alpha x) \exp(-\beta'' z) \times [\beta' \sin(\beta' z - \omega t) - \beta'' \cos(\beta' z - \omega t)] \tag{21b}$$

$$B_z = \partial A/\partial x = \alpha|A_0| \operatorname{sech}^2(\alpha x) \exp(-\beta'' z) \cos(\beta' z - \omega t). \tag{21c}$$

Focusing our attention solely on a radiation field, in the derivation of (21a) we have ignored any contribution from the scalar potential. It will be worth emphasizing that these expressions of the field profiles contain an interesting feature. In (21a) and (21b), first we note that the intensity of $E_y$ and $B_x$ has a dip around the center [Fig. 1(a)], whereas that of $B_z$ has a peak there [Fig. 1(b)]. Thus, in the vicinity of the center $(x \sim 0)$, the longitudinal field component can relatively be dominant, and the soliton becomes more "bright." The dominance is reversed as the transverse site is distant from the center.

## V. Conclusion

We have shown that a type-II superconductor may support complex solitons of weak electromagnetic radiation. They could be regarded as a solitonic version of the complex modes that were predicted for certain kinds of electromagnetic waveguides. The results presented herein will be useful for exploiting the fundamental kinetics of short-range solitons in a future microwave device that includes superconductors and/or superconductive thin films.

## ACKNOWLEDGMENT

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# On the Computation of Complex Modes in Lossless Shielded Asymmetric Coplanar Waveguides

Khwaja M. Rahman and Cam Nguyen

***Abstract*—We compute complex modes in lossless shielded asymmetric coplanar waveguides (CPW's) using the spectral domain technique. The slot asymmetry is found to significantly affect the existence of the complex modes. These modes are found to exist at low microwave frequencies even when using materials with a low permittivity. We found that waveguide modes degenerate into complex modes more frequently than CPW ($\pi$) and slotline ($c$) modes. When the structures are highly asymmetrical and when the dielectric substrates are thick or have a high permittivity, the degeneration of lower-order $c$-modes into complex modes is detected. Other forms of mode conversion, where a waveguide mode is converted to a $c$-mode, are also observed, especially in highly asymmetric structures and when using dielectric materials of a high permittivity or of a large thickness. Numerical convergence of the complex modes' propagation constants is also examined.**

## I. INTRODUCTION

Since its discovery in 1969 by C. P. Wen [1], coplanar waveguide (CPW) has been used widely for microwave integrated circuits (MIC's) and monolithic microwave integrated circuits (MMIC's) [2]. Among CPW's, the asymmetrical version is very attractive since it can provide additional circuit design flexibility and improved characteristic impedance range. Various dynamic [3] and quasi-static [4] analyses have been performed for asymmetric CPW's. However, the analysis of complex modes in asymmetrical CPW's has not yet been addressed. As will be seen, the existence of complex modes is highly pronounced in asymmetrical CPW's. They have been found even at low microwave frequencies and in low permittivity substrates. A thorough knowledge of these complex modes is thus very important for the accurate design of MIC's and MMIC's using asymmetric CPW's in both low and high microwave regions.

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In the past several years, complex modes in lossless waveguiding structures have been studied by a number of researchers. The presence of complex modes in lossless waveguiding structures was first predicted for a circular dielectric-loaded waveguide [5]. Later, theoretical as well as experimental investigations were made on the circular dielectric-loaded waveguide to confirm the existence of these modes [6], [7]. Complex modes were also reported for lossless finlines [8] and shielded microstrip lines [9], [10].

We present in this paper an extensive investigation of complex modes in lossless shielded three-layer asymmetric CPW's using the spectral domain approach (SDA) [11]. The effects of slot asymmetry, and dielectric constant and thickness of dielectric materials on the possible existence of complex modes, are described. Special attention is given to the numerical convergence of the calculated complex modes' propagation constants. The developed analysis has been applied to a symmetric CPW, and generated numerical results of the propagation constants of several real modes agree well with previously published data [12]. It should be noted here that our considered three-layer asymmetric CPW's are general in that they are applicable to both open and shielded CPW's, both symmetry and asymmetry in slots and ground planes, with and without a back-side conductor, with and without dielectric overlay, and with finite- and infinite-extent substrates. They can elude the energy leakage or increase the single-mode operating range with properly chosen dielectric substrates [13]. It is therefore expected that these asymmetrical CPW structures can be exploited to achieve MIC's and MMIC's with enhanced performance and smaller size.

## II. NUMERICAL RESULTS AND DISCUSSIONS

Complex modes in a lossless shielded three-layer asymmetric CPW's with assumed infinitesimally thin metallization (Fig. 1) are investigated using the SDA. Applying the SDA produces a system of homogeneous linear equations. By setting the determinant of the coefficient matrix of the resultant equations to zero, we can solve for the propagation constants, $\gamma$, of all of the eigenmodes. The values of $\gamma$ will be searched in the complex plane, owing to the fact that it is complex for complex modes. Due to the asymmetry in the structure, both the CPW ($\pi$) and slotline ($c$) modes will be excited along with the waveguide modes, leading to the possible existence of complex $\pi$-, $c$-, and waveguide modes. These complex modes appear in pairs and are formed when two evanescent modes degenerate into a pair of modes having $\gamma = \alpha \pm j\beta$, and which propagate in the $\pm$ directions with a phase constant $\beta$ and attenuation constant $\alpha$. These respective waves attenuate and grow exponentially with $\alpha$ as they propagate, leading to no corresponding transmitted power. Their existence is noticed when the root of the eigenvalue equation is complex, in spite of the lossless transmission line assumption.